Ode to a Big-Ass Drop of H_2O

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The take-home message, if you don't feel like reading all the blah-blah below, is that volume increases a hell of a lot faster than radius as the size of an object increases. This has a lot of quirky ramifications, one of which pertains to the densities of black holes.

In the simplest terms, a black hole is a region of space for which the escape velocity is equal to or greater than c, the speed of light. Think of a black hole as a bubble that surrounds a lump of matter. Outside the bubble, the escape velocity is less than c; on the bubble's surface, the escape velocity is equal to c; and inside the bubble, the escape velocity is greater than c.

Before talking about black holes, I'll summarize the relevant formulas and provide some convenient constants…

$F = ma = \frac{GM_1 m_2}{r^2}$	Gravitational Force G = Gravitational Constant M_1 = Big Mass (e.g. a planet) m_2 = Little Mass (e.g. a person) r = Distance between the centers of M1 & m2. If the two masses are drastically different, such as a person and a planet, then the small mass can usually be dropped from the equation. You can't do this if you are calculating the attraction between the Earth and the Moon, however.

G = 6.6726×10^{-11} $m^3/Kg\text{-}sec^2$
Earth radius = 6371 Km = 6.371×10^{6} meters
Earth mass = 5.9742×10^{24} Kg

We tend to measure gravity in terms of "g-force". For example, my car will take approx. 0.92 lateral g's before wiping out on a turn, meaning I can experience almost the equivalent of one full Earth gravity sideways before losing control. Technically, this is a measure of acceleration, not force. One Earth gravity (1 g) is about 9.8 m/sec_2.

If F = ma, then 1 g-force = $a = \frac{GM_1}{r^2} = 9.821 \frac{m}{\sec^2}$

So what the Hell is "Escape Velocity"?

Escape velocity:

- An object launched from the surface of a body has kinetic energy.
- An object at rest within the gravitational influence of a body has no kinetic energy but has some potential energy.
- An object at rest outside of the gravitational influence of the body from which it was launched has zero potential energy. This occurs only at r = infinity.

When an object is launched (thrown) upward from the surface of the planet, it will slow down, stop, and eventually fall back to Earth. The more force is used in the initial throw, the higher the object will go before falling back to Earth. If the initial force is strong enough, however, the object will never return.

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The escape velocity is the velocity required to launch an object from a body with enough kinetic energy that it doesn't come to rest until r = infinity. In other words, when the kinetic energy is depleted and the object comes to rest, it must have escaped the gravitational influence of the original body.

The equation for this escape velocity may be derived by setting the difference between the kinetic energy (K.E.) and potential energy (P.E.) to zero …

$$E = 0 = \text{K.E.} - \text{P.E.} = \frac{1}{2}mv^2 - \frac{GM_1m_2}{r}$$

For Earth, the escape velocity is:

$$\frac{1}{2}mv^2 = \frac{GM_1m_2}{r}$$

The mass of the object being launched drops out of the equation, yielding:

$$\frac{1}{2}v^2 = \frac{GM_1}{r};\ \ v_{escape} = \sqrt{\frac{2GM_1}{r}} = 11{,}186.6\ {}^{m}\!/\!_{s}$$

(answer = 11.2 Km/s = 6.95 miles/sec = 25,024 miles/h).

How small must the Earth be for the escape velocity to be *c* (the speed of light)?

Just substitute *c* for *v* and solve for the radius…

<u>Earth as black-hole</u>: $\frac{1}{2}v^2 = \frac{GM_1}{r} \Rightarrow \frac{1}{2}c^2 = \frac{GM_1}{r};\ \ r = \frac{2GM_1}{c^2}$ = 0.00887 m = 0.887 cm.

(Note: 2GM = 7.9727 X 10^{14} m^3/sec^2)

For problems involving an escape velocity of the speed of light, much of the math can be simplified. Two handy methods are (1) express velocity as a fraction of *c*, and (2) use the Gravitational constant to convert mass measurements to meters. When you do these things, units disappear and the math becomes much simpler …

Escape velocity is: $\frac{1}{2}mv^2 = \frac{GM_1m_2}{r};\ \ \frac{1}{2}v^2 = \frac{GM_1}{r}$

<u>Simplification of the math</u>:

Express velocity as a fraction of *c*: $v_f = \frac{v}{c} = 1$

Express mass in meters: $M = \frac{G}{c^2}M_1 = 0.00444_{meters} = 0.44_{cm}$

$$\frac{1}{2}v^2 = \frac{GM_1}{r} \text{ then becomes } \frac{1}{2}v_f^2 = \frac{M}{r};\ \ v_{escape} = \sqrt{\frac{2M}{r}} = 1$$

$$v_{escape} = 1 = \sqrt{\frac{2\,X\,0.00444_{meters}}{0.00888_{meters}}}$$

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Shrink the Sun to a Black Hole

Schwarzschild Radius = 2M = 2 X 1.477 Km = 2.954 Km.

Use the same math as was used for the Earth. Note that an Earth mass black hole had a radius of less than 0.9 cm, while a solar mass black hole has a radius of just less than 3 Km.

Water Drop Black Hole

How much water do you need to create a black hole, assuming no compression takes place and that your giant drop of water has a uniform density of 1 gram per cc (i.e. typical water density).

The density of an object is simply the mass divided by the volume. For a black hole, the mass and radius are related by r = 2M. [both radius and mass expressed in meters]. Therefore, one can substitute r/2 for the mass in the density equation, simplifying the formula:

Black Hole Density = $\dfrac{M}{\frac{4}{3}\pi r^3} = \dfrac{\frac{1}{2}r}{\frac{4}{3}\pi r^3} = \dfrac{3}{8\pi r^2}$

Water Density = $\dfrac{1g}{1cc} = 1000\,kg/m^3$. Next, convert kg to meters:

$$1000\,kg/m^3 \times \frac{6.6726\times10^{-11}\,m^3/kg\cdot\sec^2}{(3\times10^8\,m/\sec)^2} = 7.414\times10^{-25}\,\frac{meters}{meters^3}$$

Radius of water-hole … $7.414\times10^{-25}\,meters^{-2} = \dfrac{3}{8\pi r^2}$

Solving for r gives 4.0125 X 10^{11} meters (22.3 light minutes)

This is the radius of the water drop. Such a drop would engulf the inner solar system, including Mars. You can calculate the mass if you want, but suffice it to say, it is a BIG DAMN DROP!

By my calculations, this water drop is 2.71 x 10^{38} Kg, or about 1.36 x 10^8 Solar masses (i.e. 136 million times the mass of the Sun). For comparison, the core of galaxy M87 is approximately 3 x 10^9 Solar masses (3 billion), while the full M87 galaxy is approximately 3 x 10^{18} Solar masses.

Is the Water Drop Real?

So, what would happen, really, if you tried to condense such a ridiculously large quantity of water into one giant drop? Let's say the water starts as a uniform sphere with a radius of ten light years and contracts from there. It wouldn't contract uniformly, so very quickly a nucleus of ice would form at the center. The nucleus would grow to planet size and probably liquefy due to the heat generated from gravitational contraction. This body would continue to grow until hydrogen fusion begins, resulting in a star. As more water falls in, the body would undergo further gravitational collapse, first to a neutron star, and finally to a black hole. Thus, a smaller (but much denser) black hole would form originally. Assuming no water is lost due to the many shock waves that would erupt while all this is going on, the size of the initial black hole would continue to grow with the incoming water. Ultimately, the final black hole would have a radius of 22.3 light minutes and an average density of one gram per milliliter. Of course, this is only an *average* density – the black hole would, in theory, wind up as an incredibly dense central point *(singularity)* surrounded by empty space. The outside observer, however, would have no way of knowing if this were true.

<u>What's the point?</u>
Extrapolate well beyond the 22.3 light minute black hole and imagine regions on the scale of galactic cores. Imagine the visible universe itself. On such scales, surprisingly low densities qualify as black holes. Entire regions of space containing millions of stars within dense galactic cores may qualify as black holes. The universe itself may be a black hole. The point is simply that black holes need not be dense!